Compositionality for Markov Reward Chains with Fast Transitions

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Outline

- Introduction
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  - Discontinuous Markov reward chains
    - Ordinary lumping
    - Reduction
  - Markov reward chains with fast transitions
    - τ-lumping
    - τ-reduction
  - Relational properties
- Parallel composition

Markov Reward Chains

- Among most important and wide-spread analytical performance models
- Ever growing complexity of Markov reward chain systems
- Compositional generation: Composing a big system from several small components
- State space explosion: Result size is product of sizes of components
- Need aggregation methods...
- ...and they should be compositional
- We consider special models of Markov reward chains: Discontinuous Markov reward chains and Markov reward chains with fast transitions

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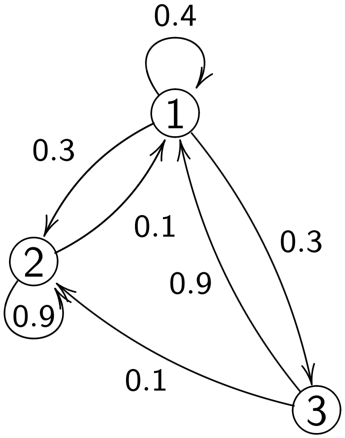
Recapitulation: Discrete time Markov chains

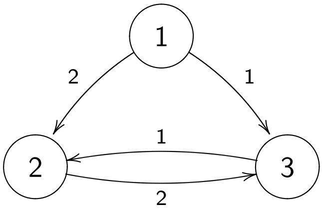


Transition probability matrix:

	1	2	3
1	0.4	0.3	0.3
2	0.1	0.9	0
3	0.9	0.1	0

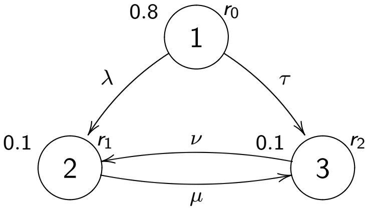
- Graphs with nodes representing states
- Outgoing arrows determine stochastic behavior of each state
- Probabilities only depend on current state

Continuous time Markov chains



- Generator matrix: $\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & -3 & 1 & 2 \\ 2 & 0 & -2 & 2 \\ 3 & 0 & 1 & -1 \end{array} = Q$

Continuous time Markov reward chains



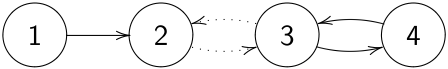
- $P = (\sigma, Q, \rho)$
- σ is a stochastic row initial probability vector $(0.8, 0.1, 0.1)$
- ρ is a state reward vector (r_0, r_1, r_2)
- Transition probability matrix

$$P(t) = \sum_{n=0}^{\infty} \frac{Q^n t^n}{n!} = e^{Qt}$$

- Rewards are used to measure performance (application dependent).

Discontinuous Markov reward chains

- Markov chains with instantaneous transitions $\rightarrow$ discontinuous Markov chains
- Discontinuous Markov reward chain: $P = (\sigma, \Pi, Q, \rho)$
- Intuition: $\Pi[i, j]$ denotes probability that a process occupies two states via an instantaneous transition.
- $\Pi = I$ leads to a standard Markov chain $\rightarrow$ generalization



Discontinuous Markov reward chains

Aggregation for discontinuous Markov reward chains

- Ordinary lumping
- Reduction

Ordinary lumping

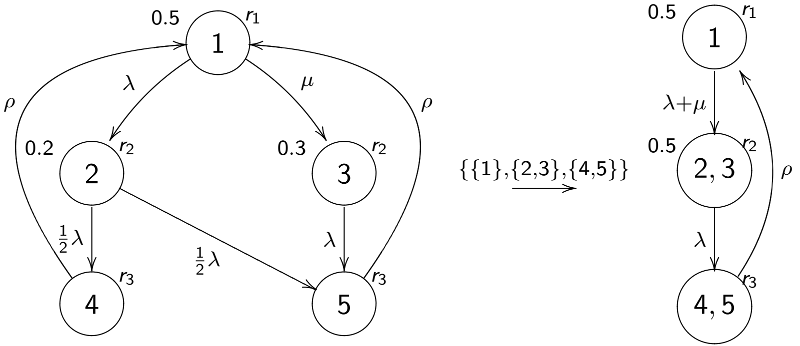
- We lump $P = (\sigma, \Pi, Q, \rho)$ to $\overline{P} = (\overline{\sigma}, \overline{\Pi}, \overline{Q}, \overline{\rho})$
- Partition $\mathcal{L}$ is an ordinary lumping
- $P \xrightarrow{\mathcal{L}} \overline{P}$

Ordinary lumping

- $P \xrightarrow{\mathcal{L}} \overline{P}$
- Partition of the state space into classes
- States lumped together form a class
- Equivalent transition behavior to other classes (intuitively: probability of class is sum of probabilities of states)
- All states in a class have the same reward, total reward is preserved

Example

- $P \xrightarrow{\mathcal{L}} \overline{P}$



Reduction

- We reduce $P = (\sigma, \Pi, Q, \rho)$ to $\overline{P} = (\overline{\sigma}, I, \overline{Q}, \overline{\rho})$
- $P \rightarrow_r \overline{P}$
- Result is unique up to state permutation.
- Canonical product decomposition of Π
- Reduced states are given by ergodic classes of the original process (ergodic = each state can be reached from each other state in finite time)
- Total reward is preserved

Markov reward chains with fast transitions

Markov reward chains with fast transitions

- Definition
- Aggregation

Markov reward chains with fast transitions

- Adds parameterized ("fast") transitions to a standard Markov reward chain.
- Uses two generator matrixes Q_s and Q_f, for slow and fast transitions.
- $P = (\sigma, Q_s, Q_f, \rho)$ is a function...
- ...where to each $\tau > 0$ a Markov reward chain $P_\tau = (\sigma, I, Q_s + \tau Q_f, \rho)$ is assigned
- The limit $\tau \to \infty$ makes fast transitions instantaneous, and we end up with a discontinuous Markov reward chain.

Markov reward chains with fast transitions

Aggregation for Markov reward chains with fast transitions

- τ-lumping
- τ-reduction

τ-lumping

- We τ-lump $P = (\sigma, Q_s, Q_f, \rho)$ to $\overline{P} = (\overline{\sigma}, \overline{Q}_s, \overline{Q}_f, \overline{\rho})$
- Can define it using the limiting discontinuous Markov reward chain.
- $P \overset{\mathcal{L}}{\rightsquigarrow} \overline{P}$
- Not unique

τ-lumping

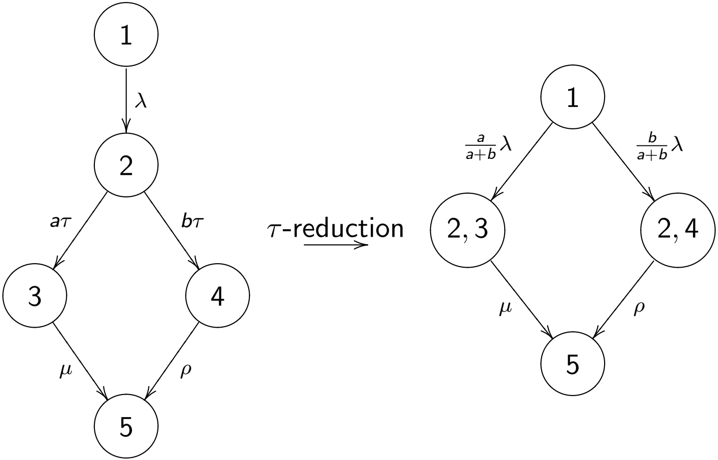
$$\begin{tikzcd} \begin{array}{c} P \\ \text{(fast transitions)} \end{array} \arrow[r, squiggly, "\mathcal{L}"] \arrow[d, "\infty"'] & \begin{array}{c} \overline{P} \\ \text{(lumped fast transitions)} \end{array} \arrow[d, "\infty"] \\ \begin{array}{c} Q \\ \text{(discontinuous)} \end{array} \arrow[r, "\mathcal{L}"] & \begin{array}{c} \overline{Q} \\ \text{(lumped discontinuous)} \end{array} \end{tikzcd}$$

τ-reduction

- We τ-reduce $P = (\sigma, Q_s, Q_f, \rho)$ to $R = (\overline{\sigma}, I, \overline{Q}, \overline{\rho})$
- $P \rightsquigarrow_r R$

Example

- $P \rightsquigarrow_r R$



τ-reduction

$$
\begin{tikzcd}
\begin{array}{c} P \\ \text{(fast transitions)} \end{array} \arrow[d, "\infty"'] \arrow[dr, rightsquigarrow, "r"'] & \\
\begin{array}{c} Q \\ \text{(discontinuous)} \end{array} \arrow[r, "r"'] & \begin{array}{c} R = R' \\ \text{(continuous)} \end{array}
\end{tikzcd}
$$

- if $P \rightsquigarrow_r R$
- and $P \rightarrow_\infty Q \rightarrow_r R'$
- then $R = R'$

Relational properties of ordinary lumping and τ-lumping

- Reduction works in one step, so no need to look at details of its relational properties.

Lumping:

- Need transitivity and strong confluence...
- ...to ensure that iterative application yields a uniquely determined process.
- Repeated application of ordinary lumping...
- ...can be replaced by single application of composition of individual lumpings.
- For τ-lumping, only the limit is uniquely determined.

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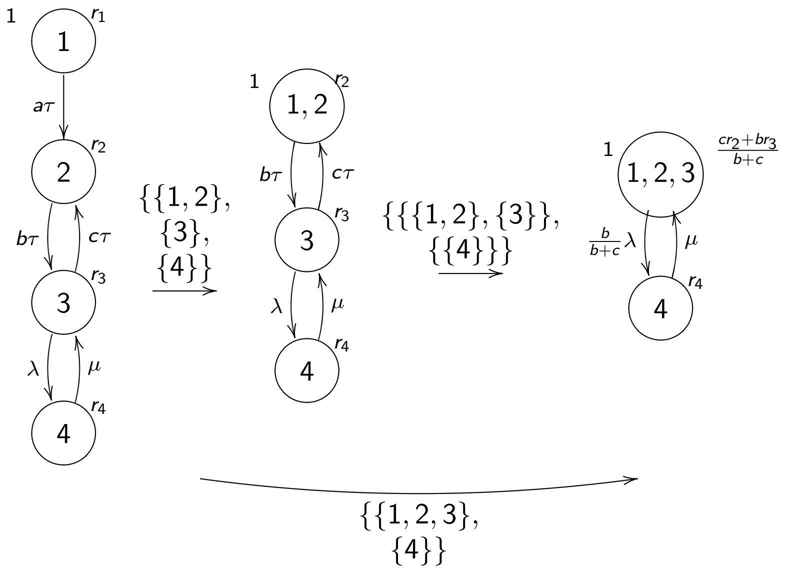


1
r_1
1
$a\tau$
r_2
2
$b\tau$
$c\tau$
r_3
3
λ
μ
r_4
4
$\{\{1,2\}, \{3\}, \{4\}\}$
1
r_2
1, 2
$b\tau$
$c\tau$
r_3
3
λ
μ
r_4
4
$\{\{\{1,2\},\{3\}\}, \{\{4\}\}\}$
1
1, 2, 3
$\frac{cr_2+br_3}{b+c}$
$\frac{b}{b+c}\lambda$
μ
r_4
4
$\{\{1,2,3\}, \{4\}\}$

Parallel composition

- $P_1 \geq \overline{P}_1, P_2 \geq \overline{P}_2 \implies P_1 \parallel P_2 \geq \overline{P}_1 \parallel \overline{P}_2$
- Aggregate smaller components first...
- ...then combine them into the aggregated complete system.
- $\geq$ is semantic preorder.
- $P \geq \overline{P}$ means $\overline{P}$ is an aggregated version of P.
- $\parallel$ is a parallel composition.

Composing discontinuous Markov reward chains

- Kronecker sum $\oplus$ and Kronecker product $\otimes$
- Parallel composition $P_1 \parallel P_2 = (\sigma_1 \otimes \sigma_2, \Pi_1 \otimes \Pi_2, Q_1 \otimes \Pi_2 + \Pi_1 \otimes Q_2, \rho_1 \otimes \mathbf{1}^{|\rho_2|} + \mathbf{1}^{|\rho_1|} \otimes \rho_2)$
- If P_1 and P_2 are discontinuous Markov reward chains, then so is $P_1 \parallel P_2$

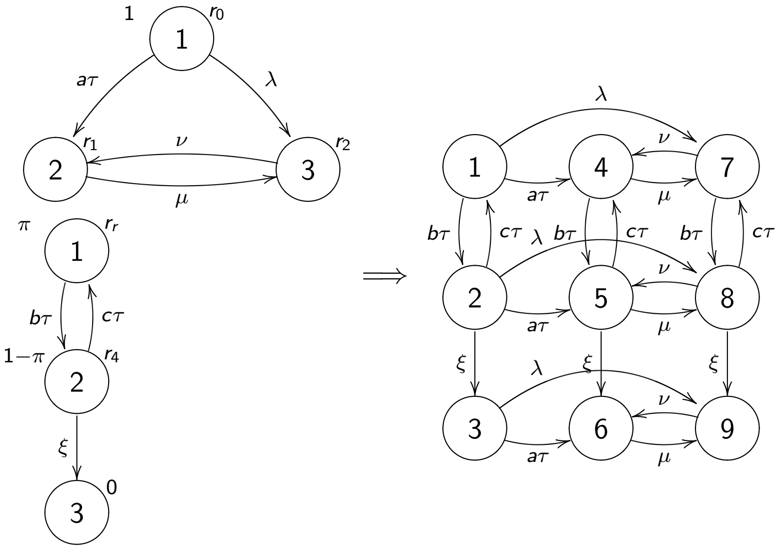
Composing discontinuous Markov reward chains

- Both lumping and reduction are compositional with respect to the parallel composition of discontinuous Markov reward chains
- If $P_1 \stackrel{\mathcal{L}_1}{\to} \overline{P}_1$ and $P_2 \stackrel{\mathcal{L}_2}{\to} \overline{P}_2$, then $P_1 \parallel P_2 \stackrel{\mathcal{L}_1 \otimes \mathcal{L}_2}{\longrightarrow} \overline{P}_1 \parallel \overline{P}_2$.
- If $P_1 \to_r \overline{P}_1$ and $P_2 \to_r \overline{P}_2$, then $P_1 \parallel P_2 \to_r \overline{P}_1 \parallel \overline{P}_2$

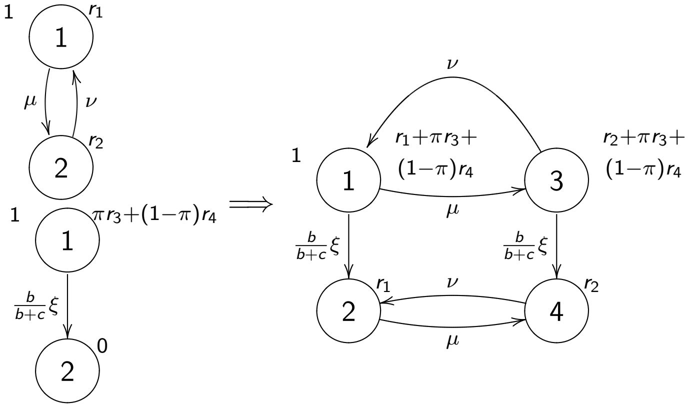
Composing Markov reward chains with fast transitions

- Parallel composition
  $P_1 \parallel P_2 = (\sigma_1 \otimes \sigma_2, Q_{s,1} \oplus Q_{s,2}, Q_{f,1} \oplus Q_{f,2}, \rho_1 \otimes \mathbf{1}^{|\rho_2|} + \mathbf{1}^{|\rho_1|} \otimes \rho_2)$
- If $P_1 \overset{\mathcal{L}_1}{\rightsquigarrow} \overline{P}_1$ and $P_2 \overset{\mathcal{L}_2}{\rightsquigarrow} \overline{P}_2$, then $P_1 \parallel P_2 \overset{\mathcal{L}_1 \otimes \mathcal{L}_2}{\rightsquigarrow} \overline{P}_1 \parallel \overline{P}_2$.
- If $P_1 \rightsquigarrow_r \overline{P}_1$ and $P_2 \rightsquigarrow_r \overline{P}_2$, then $P_1 \parallel P_2 \rightsquigarrow_r \overline{P}_1 \parallel \overline{P}_2$

Example of parallel composition



Aggregated version of composition



Summary

$$
\begin{tikzcd}
P_1 \arrow[r, rightsquigarrow, "\mathcal{L}_1"] \arrow[d, "\infty"'] & \overline{P}_1 \arrow[d, "\infty"] \\
Q_1 \arrow[r, "\mathcal{L}_1"] & \overline{Q}_1
\end{tikzcd}
\quad
\begin{tikzcd}
P_2 \arrow[r, rightsquigarrow, "\mathcal{L}_2"] \arrow[d, "\infty"'] & \overline{P}_2 \arrow[d, "\infty"] \\
Q_2 \arrow[r, "\mathcal{L}_2"] & \overline{Q}_2
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
P_1 \parallel P_2 \arrow[r, rightsquigarrow, "\mathcal{L}_1 \otimes \mathcal{L}_2"] \arrow[d, "\infty"'] & \overline{P}_1 \parallel \overline{P}_2 \arrow[d, "\infty"] \\
Q_1 \parallel Q_2 \arrow[r, "\mathcal{L}_1 \otimes \mathcal{L}_2"] & \overline{Q}_1 \parallel \overline{Q}_2
\end{tikzcd}
$$

$$
\begin{tikzcd}
P_1 \arrow[d, "\infty"'] \arrow[dr, rightsquigarrow, "r"'] & \\
Q_1 \arrow[r, "r"'] & R_1
\end{tikzcd}
\quad
\begin{tikzcd}
P_2 \arrow[d, "\infty"'] \arrow[dr, rightsquigarrow, "r"'] & \\
Q_2 \arrow[r, "r"'] & R_2
\end{tikzcd}
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End

- Questions?